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A Finite-Difference Method for Generalized Radial Transport Equations*

AVRON DOUGLIS

U.S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland 20910

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INTRODUCTION

For singular integro-differential equations including time-dependent transport equations with spherical symmetry, we shall consider solutions that assume arbitrary, prescribed initial data and satisfy boundary conditions appropriate to a bare reactor. Our primary aim is to justify a certain mode of calculation of these solutions. While the existence of the solutions follows from the proof of convergence of the calculation scheme, pure existence is demonstrable with greater ease by an iterative method, such as was used in [*4*]. Uniqueness and other properties of the solutions have already been treated in [*3*], to which we shall sometimes refer below.

This paper is parallel in some respects to a previous treatment of multi-dimensional generalized transport equations [*4*], differing from it mainly because of the singularity that occurs in radial equations. Certain features of the difference scheme imitate those in a method of Keller and Wendroff [*6*].

Recent developments in analytical and numerical methods for transport equations are reviewed by Bell, Carlson, and Lathrop [*1*].

1. STATEMENT OF PROBLEM; NOTATION

Let X denote a fixed positive constant. For $0 < x < X$, $|y| < 1$, $t > 0$, we shall ask for solutions $u(x, y, t)$ of an integro-differential equation of the form

$$u_t + yu_x + \frac{1 - y^2}{x} u_y + c(x, y, t)\, u = g(x, y, t) + Su(x, y, t), \quad (1.1)$$

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where

$$Su(x, y, t) = \int_{-1}^{1} K(x, y, t, y')\, u(x, y', t)\, dy'.$$

These solutions will be required to satisfy initial conditions of the form

$$u(x, y, 0) = \phi(x, y) \tag{1.2}$$

and the boundary condition

$$u(X, y, t) = 0 \qquad \text{when} \qquad -1 \leqslant y < 0, \quad t > 0. \tag{1.3}$$

We shall restrict our considerations to a fixed parallelepiped

$$S_T : 0 < x \leqslant X, \qquad |y| \leqslant 1, \qquad 0 \leqslant t \leqslant T,$$

with arbitrary positive T, on which c and g are assumed to be defined and u is desired. K is assumed to be given on a corresponding four-dimensional parallelepiped

$$\Sigma_T : 0 < x \leqslant X, \qquad |y| \leqslant 1, \qquad 0 \leqslant t \leqslant T, \qquad |y'| \leqslant 1,$$

and ϕ on the two-dimensional base

$$S_0 : 0 < x \leqslant X, \qquad |y| \leqslant 1,$$

of S_T.

We shall generally make at least the following assumptions:

(i) c is bounded and measurable in S_T.

(ii) g is bounded and measurable in S_T.

(iii) (a) K is integrable over Σ_T,

(b) $K \geqslant 0$,

(c) For each point (x, y, t) of S_T, $K(x, y, t, y')$ is integrable with respect to y', and

$$\int_{-1}^{1} K(x, y, t, y')\, dy' \leqslant k_0,$$

where k_0 is a constant independent of x, y, t.

(iv) ϕ is bounded and measurable over S_0.

Under almost these assumptions, we shall see that the problem stated has a "weak" solution in the following sense:

DEFINITION. A bounded, measurable function $u(x, y, t)$ is a *weak solution* of (1.1), (1.2), (1.3) if

$$\iint_{t>0} \left\{ u \left(v_t + y v_x + \left(\frac{1-y^2}{x} v \right)_y \right) + v(-cu + g + Su) \right\} dx\, dy\, dt + \int v(x, y, 0)\, \phi(x, y)\, dx\, dy = 0 \tag{1.4}$$

for any continuous, piecewise-differentiable function $v(x, y, t)$ with support in the region

$$\begin{aligned} &0 \leqslant t \leqslant T - \delta, && \\ &\delta \leqslant x \leqslant X - \delta, && \text{when} \quad y \geqslant 0, \\ &\delta \leqslant x \leqslant X, && \text{when} \quad y < 0, \\ &|\, y\,| \leqslant 1, && \end{aligned}$$

δ being an arbitrary positive number less than $\min(X, T)$.

We shall also see that a weak solution may exist when c and g are not bounded.

Under Conditions (i)–(iv), a weak solution is unique (see Theorem 4.1 and its corollary in [3]), and we know *a priori* (see [3], Section 6) that a weak solution is smooth under appropriate conditions on c, g, K, ϕ. Hence, to produce a weak solution, rather crude processes are acceptable that may give by themselves little or no information concerning the smoothness of the result. Our calculation process is of this type. It demands that any given problem, first of all, be approximated by a problem of special type. This is discussed in Section 2. Problems of the special type are then solved by finite-difference schemes, which are described in Sections 3 and 4 and, in Section 5, are proved to converge. The estimates underlying the proof of convergence are developed in Sections 6, 7, and 8.

2. Problems with Truncation

We describe g as "truncated" if, for a positive constant ω,

$$g(x, y, t) = 0 \qquad \text{for} \quad |\, y\,| < \omega.$$

We call K truncated if

$$K(x, y, t, y') = 0 \qquad \text{for} \quad |\, y\,| < \omega.$$

Only when g and K are truncated have we been able to justify our finite-difference scheme. As we shall show later in this section the solutions of a wide class of problems without truncation are the limits of solutions of appropriate problems with truncated g and K.

For a point set A, let $V^+(A)$ denote the class of functions w, defined and finite almost everywhere (a.e.) on A, each of which is the limit of a monotonically increasing sequence of bounded, Lipschitz-continuous functions in A. Let $V^{+-}(A)$ denote the class of functions W on A, each of which is finite almost everywhere and is the limit of a monotonically decreasing sequence of functions belonging to $V^+(A)$. Any bounded, measurable function A is equal a.e. to a member of $V^{+-}(A)$ (this is easily seen from Graves [5], p. 217).

We shall be able to approximate the solutions of problems that satisfy the following assumptions, in addition to (iii):

(i)$_+$ $\quad c \geqslant c_0 \quad (c_0 = \text{constant}), \quad -c \in V^+(S_T)$.

$(\overline{\text{ii}})$ $\quad g = g_1 - g_2, \quad \text{where} \quad g_i \geqslant 0, \quad g_i \in V^+(S_T), \quad i = 1, 2.$

(iii)$_+$ $\quad K \in V^+(\Sigma_T)$.

$(\overline{\text{iv}})$ $\quad \phi = \phi_1 - \phi_2, \quad \text{where} \quad \phi_i \geqslant 0, \quad \phi_i \in V^+(S_0), \quad i = 1, 2.$

To be able to approximate the solution u of a problem in which these assumptions are satisfied, it suffices to be able to approximate the solutions u_i of the two related problems in which g_i and ϕ_i, $i = 1, 2$, take the places of g and ϕ, respectively. This follows from the superposition principle, since $u = u_1 - u_2$. Therefore, we do not lose generality in specializing assumptions $(\overline{\text{ii}})$ and $(\overline{\text{iv}})$ to the following:

(ii)$_+$ $\quad g \geqslant 0, \quad g \in V^+(S_T)$;

(iv)$_+$ $\quad \phi \geqslant 0, \quad \phi \in V^+(S_0)$.

In addition, since $c \geqslant c_0$, by making the change of variable $v = u\, e^{c_0 t}$, we are led to an equation for which the normalized condition,

(i)$_N$ $\quad c \geqslant 0,$

holds. It follows from (ii)$_+$ and (iii)$_+$ that

> (Trunc) g and K are the limits of monotonically increasing sequences of bounded, Lipschitz-continuous, *nonnegative*, *truncated* functions, respectively.

In fact, g and K are limits of certain monotonically increasing sequences, respectively, whose members can be assumed to be nonnegative. Then those members can be truncated in such a way as to preserve nonnegativity, Lipschitz continuity, monotonicity, and convergence to the stipulated limits. A feasible method of truncation will be given below.

Let us describe a problem as truncated and regular if the functions c, g, ϕ, K entering into this problem are nonnegative, bounded, and Lipschitz-continuous and if, in addition, g and K are truncated. To any problem satisfying the hypotheses

$$(\mathrm{i})_+, \qquad (\mathrm{i})_\mathrm{N}, \qquad (\mathrm{ii})_+, \qquad (\mathrm{iii}), \qquad (\mathrm{iii})_+, \qquad (\mathrm{iv})_+, \tag{2.1}$$

[with condition (Trunc) also holding because of $(\mathrm{ii})_+$ and $(\mathrm{iii})_+$] corresponds a sequence of truncated, regular problems, the functions of which tend monotonically to the corresponding functions c, g, K, ϕ, originally prescribed. We shall see that the solutions u_k, $k = 1, 2,...$, of the truncated, regular problems are Lipschitz-continuous and uniformly bounded, and increase monotonically with k. It will follow that $u = \lim_{k\to\infty} u_k$ exists, that $u \in V^+(S_T)$, and by going to the limit under the sign of integration that u is a weak solution of the original problem. Thus, we will have this result:

THEOREM 2.1. *The weak solution u of a problem satisfying conditions* $(\mathrm{i})_+$, $(\mathrm{i})_\mathrm{N}$, $(\overline{\mathrm{ii}})$, (iii), $(\mathrm{iii})_+$, $(\overline{\mathrm{iv}})$ *can be represented as*

$$u = \lim_{k\to\infty} (v_k - w_k),$$

where v_k and w_k are the solutions of appropriate truncated regular problems.

A similar result can be based upon the minimal assumptions of Section 1, in addition to

$(\mathrm{iii})_{+-}$ $\qquad K \in V^{+-}(\Sigma_T)$.

THEOREM 2.2. *The weak solution u of a problem satisfying conditions* (i), $(\mathrm{i})_\mathrm{N}$, (ii), (iii), $(\mathrm{iii})_{+-}$, (iv) *can be represented as the limit* a.e. *in S_T of a sequence of differences*

$$v_k - w_k, \qquad k = 1, 2,...,$$

where v_k and w_k are solutions of truncated, regular problems.

Proof. Since $-c$ is bounded and measurable, $-c$ is equal almost everywhere to a function we shall again denote by $-c$ belonging to $V^{+-}(S_T)$. Since g is bounded and measurable, we have

$$g = g_1 - g_2 \qquad \text{a.e.},$$

where $g_i \geqslant 0$, and g_i is bounded and measurable, $i = 1, 2$. Each g_i is equivalent to a function, which we again shall denote by g_i, belonging to $V^{+-}(S_T)$. Similarly,

$$\phi = \phi_1 - \phi_2 ,$$

where $\phi_i \geqslant 0$ and we may assume that $\phi_i \in V^{+-}(S_0)$, $i = 1, 2$. If u_i denotes the weak solution of the problem, with g_i and ϕ_i in place of g and ϕ, respectively, then $u = u_1 - u_2$ is a weak solution of the original problem. Thus, Theorem 2.2 will follow from the special case in which the following assumptions hold in addition to $(\mathrm{i})_{\mathrm{N}}$, (iii), and $(\mathrm{iii})_{+-}$:

$(\mathrm{i})_{+-}$ $-c$ is bounded and belongs to $V^{+-}(S_T)$;
$(\mathrm{ii})_{+-}$ g is bounded, $g \geqslant 0$, and $g \in V^{+-}(S_T)$;
$(\mathrm{iv})_{+-}$ ϕ is bounded, $\phi \geqslant 0$, and $\phi \in V^{+-}(S_0)$.

Proving the following lemma thus completes the proof of Theorem 2.2.

LEMMA. *The weak solution u of a problem satisfying conditions* $(\mathrm{i})_{\mathrm{N}}$, $(\mathrm{i})_{+-}$, $(\mathrm{ii})_{+-}$, (iii), $(\mathrm{iii})_{+-}$, $(\mathrm{iv})_{+-}$ *can be represented as the limit almost everywhere in S_T of solutions*

$$u_k , \qquad k = 1, 2, \ldots,$$

of truncated, regular problems.

Proof. By hypothesis, sequences of functions

$$-c_k \in V^+(S_T), \qquad g_k \in V^+(S_T), \qquad K_k \in V^+(\Sigma_T), \qquad \phi_k \in V^+(S_0)$$

exist such that

$$c_k \geqslant 0, \qquad g_k \geqslant 0, \qquad K_k \geqslant 0, \qquad \phi_k \geqslant 0$$

and

$$-c_k \searrow -c, \qquad g_k \searrow g, \qquad K_k \searrow K, \qquad \phi_k \searrow \phi \qquad \text{as} \quad k \to \infty.$$

We have seen in the proof of Theorem 2.1 that the problem that results from attaching the subscript k to each of the quantities c, g, K, ϕ has a solution U_k belonging to $V^+(S_T)$ and obtainable as

$$U_k = \lim_{p \to \infty} U_{kp} \quad \text{in} \quad S_T ,$$

where U_{kp} for $p = 1, 2, \ldots$ is the solution of a truncated, regular problem. The U_k are uniformly bounded ([3], Theorem 4.1) and decrease monotonically as $k \to \infty$ ([3], Theorem 5.3). Hence, the U_k converge on S_T to a limit u:

$$u = \lim_{k \to \infty} U_k \quad \text{in} \quad S_T ,$$

which, by Lebesgue's Theorem, is a weak solution. Since pointwise convergence on a compact set implies convergence in measure, we thus have, in particular,

$$u = \text{limit in measure of } U_k \quad \text{as} \quad k \to \infty$$

and

$$U_k = \text{limit in measure of } U_{kp}, \quad \text{as} \quad p \to \infty,$$

from which we conclude that a sequence of indices p_k, $k = 1, 2,\ldots$, exists such that

$$u = \text{limit in measure of } U_{kp_k}, \quad \text{as} \quad k \to \infty.$$

Since a sequence converging in measure has a subsequence that converges a.e., an infinite sequence (k') of the integers exists such that

$$u = \lim_{k' \to \infty} U_{k', p_{k'}} \quad \text{a.e. in} \quad S_T.$$

Now if u_1 denotes the solution $U_{k', p_{k'}}$ with smallest k', u_2 that with second smallest k', etc., then $u_1, u_2,\ldots$, are solutions of truncated regular problems having u as limit almost everywhere, as required.

An adequate truncation procedure has been described in [*4*], Section 10. It is based on a monotonic, infinitely differentiable function Φ such that

$$\begin{aligned} \Phi(s) &= 0 \qquad \text{for} \quad 0 \leqslant s \leqslant \tfrac{1}{2} \\ &= 1 \qquad \text{for} \quad s \geqslant 1. \end{aligned}$$

Multiplying any function by $\Phi(2 \mid y \mid^{1/m})$ for any $m = 1, 2,\ldots$, truncates the function with respect to y. So truncating, with appropriate variation of m, the members of any monotonic sequence of bounded, Lipschitz-continuous, nonnegative functions preserves monotonicity, boundedness, Lipschitz continuity, and nonnegativity. Thus, the truncations required can be achieved by these means.

With Theorems 2.1 and 2.2 as justification, the remainder of the paper will deal with truncated, regular problems.

3. Finite-Difference Notation: Some Remarks

Let I and J be any positive integers and

$$h = X/I, \qquad h' = 1/J.$$

Let $k > 0$, and set

$$\theta = k/h, \qquad \theta' = k/h'.$$

Let $S^{h,h',k}$ denote the set of "lattice points"

$$(ih, jh', nk), \qquad i = 0, 1,..., I; \quad j = 0, \pm 1,..., \pm J; \quad k = 0, 1,... .$$

Our aim will be to approximate the solution of a truncated, regular problem by functions, which we denote by $u^{h,h',k}(x, y, t)$, defined on $S^{h,h',k}$. The forward t-differences and backward x- and y-differences of $u^{h,h',k}(x, y, t)$ we denote by $p^{h,h',k}(x, y, t)$, $q^{h,h',k}(x, y, t)$, and $r^{h,h',k}(x, y, t)$, respectively. When h, h', and k are fixed, for brevity we set

$$u_{ij}^n = u^{h,h',k}(ih, jh', nk)$$

with similar denotations for the first differences of this function. Thus,

$$\begin{aligned}
p_{ij}^n &= (u_{ij}^{n+1} - u_{ij}^n)/k \qquad \text{for} \quad n \geqslant 0, \quad 0 \leqslant i \leqslant I, \quad |j| \leqslant J, \\
q_{ij}^n &= (u_{ij}^n - u_{i-1,j}^n)/h \qquad \text{for} \quad n \geqslant 0, \quad 1 \leqslant i \leqslant I, \quad |j| \leqslant J, \\
r_{ij}^n &= (u_{ij}^n - u_{i,j-1}^n)/h' \qquad \text{for} \quad n \geqslant 0, \quad 0 \leqslant i \leqslant I, \quad 1 - J \leqslant j \leqslant J.
\end{aligned}$$

We shall also write

$$c_{ij}^n = c(ih, jh', nk), \qquad g_{ij}^n = g(ih, jh', nk),$$

and

$$b_{ij} = (1 - j^2h'^2)/(i + 1)\, h.$$

We approximate the kernel by averages:

$$K_{ijj'}^n = (hh'^2k)^{-1} \int_{\mathsf{P}_{ijj'}^n} K(x, y, t, y')\, dy'\, dx\, dt\, dy,$$

the domain of integration being the parallelepiped

$$\mathsf{P}_{ijj'}^n : \begin{cases} hi \leqslant x < h(i + 1) \\ h'j \leqslant y < h'(j + 1) \\ h'j' \leqslant y' < h'(j' + 1) \\ nk \leqslant t < k(n + 1) \end{cases} .$$

By assumption (iii),

$$K_{ijj'}^n \geqslant 0$$

[this inequality is used only in the truncation (Section 2), however, and is not necessary for the finite-difference scheme to converge)] and

$$h'\Sigma_{j'} K_{ijj'}^n \leqslant k_0 . \tag{3.1}$$

If, as we shall later assume,

$$\int_{-1}^{1} | K(x, y, t, y') - K(x', y, t, y') | \, dy' \leqslant k_1 | x - x' |$$

for $0 \leqslant x,\ x' \leqslant X,\ | y | \leqslant 1,\ 0 \leqslant t \leqslant T,\ k_1$ being a constant, then also

$$h' \Sigma_{j'} | K^n_{ijj'} - K^n_{i-1,j,j'} | \leqslant hk_1 . \tag{3.2}$$

Similar inequalities apply to the n-differences and j-differences of $K^n_{ijj'}$.

In our finite-difference scheme (Section 4), we shall assume that $K(x, y, t, y')$ is continuous at almost every point of Σ_T. The step function defined as

$$K^{h,h',k}(x, y, t, y') = K^n_{ijj'} \quad \text{in} \quad \mathsf{P}^n_{ijj'}$$

under this assumption tends to $K(x, y, t, y')$ at each of its points of continuity in Σ_T, as $h, h', k \to 0$. Hence,

$$\lim_{h,h',k \to 0} K^{h,h',k} = K \quad \text{a.e. in} \quad \Sigma_T .$$

This suggests that we attempt to approximate the integral in Eq. (1.1) by sums (essentially Riemann sums) of the form

$$S^n_{ij} \equiv h' \Sigma_{j'} K^n_{ijj'} u^n_{ij'} .$$

The attempt succeeds, for instance, when the step functions defined by

$$u^{h,h',k}(x, y, t) = u^n_{ij} \qquad \text{for} \quad \begin{cases} kn \leqslant t < k(n+1), \\ hi \leqslant x < h(i+1), \\ h'j \leqslant y < h'(j+1) \end{cases}$$

are uniformly bounded and tend to a limit $u(x, y, t)$ at almost all points (x, y, t) of S_T as $h, h', k \to 0$, the approach of each parameter to zero being through suitable values. To be more specific, let $h_m, h_m', k_m, m = 1, 2,\ldots,$ be sequences of values of $h, h', k,$ respectively, such that the functions

$$u_m = u^{h_m, h_m', k_m},$$

as stipulated, are uniformly bounded and tend to u almost everywhere in S_T, as $m \to \infty$. In terms of

$$K_m = K^{h_m, h_m', k_m},$$

the Riemann sums considered can be written as

$$\int_{-1}^{1} K_m(x, y, t, y')\, u_m(x, y', t)\, dy'.$$

The existence of the limit of these Riemann sums need be proved only in a weak sense. Let w denote a continuous function on S_T. We shall need to know only that, under the hypotheses explained, for any such w we have

$$\int_{S_T} w\, dx\, dy\, dt \int (K_m u_m - Ku)\, dy' \to 0, \tag{3.3}$$

as $m \to \infty$. To see this, we write the integrand for the inner integral as

$$K(u_m - u) + u_m(K_m - K)$$

and express the entire integral expression as the sum of two expressions, the first J_{1m} corresponding to the first summand above and the second J_{2m} corresponding to the second summand. The fact that J_{1m} and J_{2m} tend to zero as $m \to \infty$ follows from the convergence of u_m and K_m, the boundedness of u_m, inequality (3.1), and the uniform absolute continuity of the integrals $\int K_m\, dV$, where dV denotes the volume element $dy'\, dx\, dt\, dy$ in Σ_T. To be assured of the last property, let δ be any positive number and E a measurable set in S_T of Lebesgue measure less than δ. Denoting Lebesgue measure by μ, we thus have

$$\mu(E) < \delta.$$

Re-index the parallelepipeds $\mathsf{P}^n_{ijj'}$ that intersect E as

$$\mathsf{P}_\alpha, \qquad \alpha = 1, \ldots, a_m.$$

Then

$$E = \bigcup_{\alpha=1}^{a_m} \mu(E\mathsf{P}_\alpha),$$

and from the definition of K_m as a step function,

$$\int_E K_m\, dV = \sum_{\alpha=1}^{a_m} \mu(E\mathsf{P}_\alpha) \int_{\mathsf{P}_\alpha} K\, dV.$$

We thus have, in particular,

$$\int_E K_m\, dV \leqslant \mu(E) \int_{\Sigma_T} K\, dV < \delta k_0 2XT:$$

hence, the integrals $\int K_m \, dV$ are absolutely continuous uniformly with respect to m, as asserted. This was the last property needed to justify our conclusion that J_{1m} and J_{2m} tend to zero as $m \to \infty$ and thus to justify (3.3).

4. Statement of Difference Equations: Their Recursive Character

The difference equations set up below will be justified only for truncated problems satisfying hypotheses we shall now specify fully:

$(\mathrm{i})_1$ c and g are Lipschitz-continuous on S_T with uniform Lipschitz constants we shall denote by c_1 and g_1, respectively. Furthermore, constants c_0 and g_0 exist such that $| c | \leqslant c_0$ and $| g | \leqslant g_0$.

$(\mathrm{iii})_1$ (a) K is continuous almost everywhere in Σ_T.

(b) For each point (x, y, t) of S_T, $K(x, y, t, y')$ is integrable with respect to y', and

$$\int_{-1}^{1} | K(x, y, t, y') | \, dy' \leqslant k_0 ,$$

where k_0 is a constant.

(c) A constant k_1 exists such that, if $DK(x, y, t, y')$ symbolizes a difference quotient of $K(x, y, t, y')$ with respect to any of the three arguments x, y, t, e.g.,

$$\frac{K(x, y, t + \Delta t, y') - K(x, y, t, y')}{\Delta t},$$

then

$$\int_{-1}^{1} | DK(x, y, t, y') | \, dy' \leqslant k_1 .$$

$(\mathrm{iv})_1$ ϕ is Lipschitz-continuous on S_0 with a uniform Lipschitz constant we denote by ϕ_1, while for $| y | \leqslant 1$, $| y + \Delta y | \leqslant 1$,

$$| \phi(x, y + \Delta y) - \phi(x, y)| \leqslant \phi_1 \, x \, | \Delta y |.$$

In addition,

$$\phi(0, y) = \phi(0, -1),$$
$$\phi(X, y) = 0 \qquad \text{for} \quad y \leqslant 0,$$

and a constant ϕ_0 exists such that $| \phi | \leqslant \phi_0$.

(Tr) For a positive constant ω,

$$g(x, y, t) = 0 \qquad \text{for} \quad |y| < \omega.$$
$$K(x, y, t, y') = 0 \qquad \text{for} \quad |y| < \omega.$$

Our difference scheme follows. We prescribe as initial conditions

$$u_{ij}^0 = \phi(ih, jh'), \qquad 0 \leqslant i \leqslant I, \qquad |j| \leqslant J, \tag{4.1}$$

and as principal boundary condition

$$u_{Ij}^n = 0 \qquad \text{for} \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0. \tag{4.2}$$

We also impose a second boundary condition, namely,

$$u_{0j}^n = u_{0,-J}^n \qquad \text{for} \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0. \tag{4.3}$$

Both boundary conditions are satisfied automatically for $n = 0$ because of (4.1) and $(\text{iv})_1$. Our difference approximations of (1.1) depend on the sign of j. Therefore, we formulate them in two statements as

$$u_{ij}^{n+1} - u_{ij}^n + \theta jh'(u_{i+1,j}^n - u_{ij}^n) + \theta' b_{ij}(u_{ij}^{n+1} - u_{i,j-1}^{n+1}) + kc_{ij}^n u_{ij}^n = kg_{ij}^n + kS_{ij}^n$$
$$\text{for} \quad 0 \leqslant i \leqslant I - 1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0 \tag*{(4.4)$_-$}$$

and

$$u_{ij}^{n+1} - u_{ij}^n + \theta jh'(u_{ij}^n - u_{i-1,j}^n) + \theta' b_{ij}(u_{ij}^{n+1} - u_{i,j-1}^{n+1}) + kc_{ij}^n u_{ij}^n = kg_{ij}^n + kS_{ij}^n$$
$$\text{for} \quad 1 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0. \tag*{(4.4)$_+$}$$

Since $b_{i,-J} = 0$, we shall understand Eq. $(4.4)_-$ to omit the term

$$\theta' b_{ij}(u_{ij}^{n+1} - u_{i,j-1}^{n+1})$$

for the value $j = -J$.

The idea of taking the singular terms (those containing b_{ij}) at the $(n + 1)$st instead of the nth time step was adapted from Keller and Wendroff [*6*].

Equations (4.1)–(4.4) define a recursive scheme. This is perhaps more apparent after conditions (4.4) have been put into the equivalent forms:

$$(1 + \theta' b_{ij})\, u_{ij}^{n+1} = (1 + \theta jh' - kc_{ij}^n)\, u_{ij}^n - \theta jh' u_{i+1,j}^n$$
$$+ \theta' b_{ij} u_{i,j-1}^{n+1} + kg_{ij}^n + kS_{ij}^n$$
$$\text{for} \quad 0 \leqslant i \leqslant I - 1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0, \tag*{(4.5)$_-$}$$

$$(1 + \theta' b_{ij}) u_{ij}^{n+1} = (1 - \theta j h' - k c_{ij}^n) u_{ij}^n + \theta j h' u_{i-1,j}^n$$
$$+ \theta' b_{ij} u_{i,j-1}^{n+1} + k g_{ij}^n + k S_{ij}^n$$
$$\text{for} \quad 1 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0. \tag{4.5$_+$}$$

[In (4.5)$_-$, the term $\theta' b_{ij} u_{i,j-1}^{n+1}$ is omitted when $j = -J$.] Since Eqs. (4.5)$_-$ are obviously recursive for $j = -J$, we can solve for $u_{i,-J}^1$ for $0 \leqslant i \leqslant I - 1$. Then we can obtain u_{ij}^1 one by one for $1 - J \leqslant j \leqslant 0$ and all i from Eqs. (4.5)$_-$ and Conditions (4.1) and (4.2). For $1 \leqslant j \leqslant J$ and all i we can obtain u_{ij}^1 from Eqs. (4.5)$_+$ and Conditions (4.1) and (4.3). Once all u_{ij}^n for any fixed index n and for all i and j have been determined, we can then get u_{ij}^{n+1} for all i and j by a similar procedure. This shows that Eqs. (4.1)–(4.4) define a recursive scheme, as asserted.

5. Proof that Difference Scheme Converges

Under the hypotheses of Section 4, we shall now show that the difference scheme of that section converges as $h \to \infty$, giving a weak solution, provided these conditions are fulfilled:

$$k/h = \theta = \text{const}, \tag{5.1}$$

$$2\theta + k c_0 \leqslant 1, \tag{5.2}$$

$$h' \log h \to 0 \qquad \text{as} \qquad h \to 0. \tag{5.3}$$

Our proof depends upon the facts that, under stated hypotheses, $u^{h,h',k}$ is bounded in S_T and, for any constant δ such that $0 < \delta < X$, satisfies a Lipschitz condition in the region

$$S_{T,\delta} : \begin{cases} \delta \leqslant x \leqslant X \\ |y| \leqslant 1 - \delta \\ 0 \leqslant t \leqslant T \end{cases}$$

with the same bound and the same Lipschitz constant for all possible values of h, h', k fulfilling Conditions (5.1) and (5.2). A suitable bound is given below in Section 6 and Lipschitz constants with respect to t, x, and y in Sections 6–8. Because of these properties of boundedness and Lipschitz continuity, the functions $u^{h,h',k}$ are equibounded and equicontinuous in any $S_{T,\delta}$. Hence, from any infinite set of mesh widths h, h', k containing arbitrarily small values of h and satisfying (5.1) and (5.2), sequences

$$h_m, h'_m, k_m, \qquad m = 1, 2, \ldots,$$

can be selected such that the solutions

$$u_m = u^{h_m, h_m', k_m}$$

converge continuously (see Pucci [7]) to a function u that is Lipschitz-continuous in every $S_{T,\delta}$. By continuity, u satisfies Conditions (1.2) and (1.3), and we shall see that u is a weak solution of the problem. A weak solution being unique ([3], Theorem 4.1 and its corollary), it will then be clear that $u^{h,h',k}$—not merely a special sequence of these functions—converges continuously to u as $h \to 0$.

To show that u must be a weak solution of (1.1), (1.2), (1.3), let v denote a function of class C' in S_T such that

$$v(x, y, t) = 0 \qquad \text{for} \quad \begin{cases} t \geqslant T - \delta \\ 0 \leqslant x \leqslant \delta \\ X - \delta \leqslant x \leqslant X, \qquad y \geqslant 0 \end{cases}.$$

Let (h, h', k) denote any member of the sequence (h_m, h_m', k_m), $m = 1, 2, \ldots$, such that $0 < h < \delta$, and denote the values of v on $S^{h,h',k}$ by

$$v_{ij}^n = v(ih, jh', nk).$$

From $(4.4)_-$ we have

$$hh'k \sum_{n=0}^{N-1} \sum_{i=0}^{I-1} \sum_{j=-J}^{0} v_{ij}^n \left\{ \frac{u_{ij}^{n+1} - u_{ij}^n}{k} + jh' \left(\frac{u_{i+1,j}^n - u_{ij}^n}{h} \right) \right.$$

$$\left. + \frac{(1 - j^2h'^2)}{(i+1)h} \left(\frac{u_{ij}^{n+1} - u_{i,j-1}^{n+1}}{h'} \right) + c_{ij}^n u_{ij}^n - g_{ij}^n - S_{ij}^n \right\} = 0, \tag{5.4)_-$$

and from $(4.4)_+$ we have

$$hh'k \sum_{n=0}^{N-1} \sum_{i=1}^{I} \sum_{j=1}^{J} v_{ij}^n \left\{ \frac{u_{ij}^{n+1} - u_{ij}^n}{k} + jh' \left(\frac{u_{ij}^n - u_{i-1,j}^n}{h} \right) \right.$$

$$\left. + \frac{(1 - j^2h'^2)}{(i+1)h} \left(\frac{u_{ij}^{n+1} - u_{i,j-1}^{n+1}}{h'} \right) + c_{ij}^n u_{ij}^n - g_{ij}^n - S_{ij}^n \right\} = 0. \tag{5.4)_+$$

The boundary conditions imposed upon u_{ij}^n and v_{ij}^n imply that

$$v_{ij}^N, \; v_{ij}^{N-1}, \; v_{Ij}^n u_{Ij}^n, \; v_{I-1,j}^n u_{Ij}^n, \; v_{0j}^n, \; v_{1j}^n$$

are zero. Hence, eliminating u-differences from $(5.4)_-$ and $(5.4)_+$ by summations by parts and adding the results we obtain

$$hh'k \sum_{n=1}^{N-1} \sum_{i=1}^{I-1} \sum_{j=-J}^{J} \left\{ -u_{ij}^n \left(\frac{v_{ij}^n - v_{ij}^{n-1}}{k} \right) + v_{ij}^n (c_{ij}^n u_{ij}^n - g_{ij}^n - S_{ij}^n) \right\}$$

$$-hh' \sum_{i=1}^{I-1} \sum_{j=-J}^{J} \left\{ v_{ij}^0 u_{ij}^0 - k v_{ij}^0 (c_{ij}^0 u_{ij}^0 - g_{ij}^0 - S_{ij}^0) \right\}$$

$$-hh'k \sum_{n=0}^{N-1} \sum_{i=1}^{I-1} \left\{ \sum_{j=-J}^{0} jh' u_{ij}^n \left(\frac{v_{ij}^n - v_{i-1,j}^n}{h} \right) + \sum_{j=1}^{J} jh' u_{ij}^n \frac{v_{i+1,j}^n - v_{ij}^n}{h} \right\}$$

$$-hh'k \sum_{n=0}^{N-1} \sum_{i=1}^{I-1} \frac{1}{(i+1)h} \sum_{j=1-J}^{J-1} u_{ij}^{n+1} \{[1-(j+1)^2 h'^2]\, v_{i,j+1}^n - (1-j^2h'^2)\, v_{ij}^n\}/h'$$

$$-hk \sum_{n=0}^{N-1} \sum_{i=1}^{I-1} \frac{1}{(i+1)h} h'(2-h')\, v_{i,1-J}^n u_{i,-J}^{n+1} = 0, \tag{5.5}$$

the expression $h'(2-h')$ in the final summation originating from $(1-(J-1)^2 h'^2)$, since $Jh' = 1$.

The last summation on the left side of (5.5) tends to zero with h. In fact, since u_{ij}^n and v_{ij}^n are bounded, $kN = T$, and $0 < 2 - h' < 2$, this sum is less in absolute value than an appropriate constant times

$$h' \sum_{i=1}^{I-1} \frac{1}{i+1},$$

which is less than $h' \log I$. However,

$$\begin{aligned} h' \log I &= h' \log X - h' \log h \\ &\to 0, \end{aligned}$$

as h and h' tend to zero while satisfying Condition (5.3). Hence, the last summation on the left side of (5.5) tends to zero with h, as asserted. Each of the other summations tends, by continuous convergence and (3.3) (Corollary to Theorem 6.3 in [2]), to a certain definite integral as $m \to \infty$, and the result is (1.4).. This means that u is a weak solution, as contended. Since, in addition, u is Lipschitz-continuous in $S_{T,\delta}$ for $0 < \delta < X$, we conclude that u has derivatives with respect to x, y, and t, and actually satisfies Eq. (1.1), almost everywhere in S_T (see Theorem 3.2 in [3]).

6. Bounds for the Solution of the Difference Equations and for its t-Difference Quotients

In this section, under the hypotheses of Section 4 and the condition

$$1 - \theta - kc_0 \geqslant 0, \tag{6.1}$$

we shall first prove that

$$| u^{h,h',k}(x, y, t)| < M(t), \tag{6.2}$$

where

$$M(t) = \phi_0 e^{(c_0+k_0)t} + \frac{g_0}{c_0 + k_0}(e^{(c_0+k_0)t} - 1).$$

Under the same assumptions, we shall then obtain a Lipschitz condition with respect to t of the form

$$| p^{h,h',k}(x, y, t)| \leqslant M_1(T) \qquad \text{for} \quad 0 \leqslant t \leqslant T, \tag{6.3}$$

where M_1 depends only upon T and the constants in the hypotheses of Section 4.

For $0 \leqslant nk \leqslant T$, set

$$M_n = \max | u_{ij}^n |,$$

the maximum being taken for $0 \leqslant i \leqslant I$, $| j | \leqslant J$. The coefficient of u_{ij}^n in the right member of (4.5)$_-$ is positive by assumption (6.1); the coefficient of $u_{i+1,j}^n$ in the same expression also is positive. Since the conditions $\alpha \geqslant 0$, $\beta \geqslant 0$, and $| u | \leqslant M$, $| v | \leqslant M$ imply that $| \alpha u + \beta v | \leqslant (\alpha + \beta) M$, and since, furthermore,

$$| S_{ij}^n | \leqslant k_0 M_n ,$$

we have from (4.5)$_-$

$$(1 + \theta' b_{ij}) \, | u_{ij}^{n+1} | \leqslant (1 + k(c_0 + k_0)) \, M_n + \theta' b_{ij} M_{n+1} + kg_0 \tag{6.4}$$

for $0 \leqslant i \leqslant I - 1$, $-J \leqslant j \leqslant 0$, $n \geqslant 0$. We shall see that (6.4) actually holds for all possible values of the indices. The cases $i = I$, $-J \leqslant j \leqslant 0$, $n \geqslant 0$ follow from (4.2). The cases $i = 0$, $1 \leqslant j \leqslant J$, $n \geqslant 0$ follow when we recall (4.3) and note

$$| u_{0,-J}^{n+1} | \leqslant (1 + k(c_0 + k_0)) \, M_n + kg_0 , \tag{6.4$_-$}$$

which is a special case of (6.4). In fact, we have

$$| u_{0j}^{n+1} | = | u_{0,-J}^{n+1} | \leqslant (1 + k(c_0 + k_0)) \, M_n + kg_0,$$

(6.4) following trivially for the indicated values of j. The cases $1 \leqslant i \leqslant I$, $1 \leqslant j \leqslant J$, $n \geqslant 0$ result from $(4.5)_+$ as the first cases did from $(4.5)_-$. Hence, (6.4) holds for all possible values of i, j, n, as asserted.

Find values of i and j such that $|u_{ij}^{n+1}| = M_{n+1}$, and choose these values in (6.4). Substituting and making an obvious concellation then gives us

$$M_{n+1} \leqslant (1 + k(c_0 + k_0))\, M_n + kg_0 \qquad \text{for} \quad n \geqslant 0.$$

By induction, we then have

$$M_n \leqslant (1 + k(c_0 + k_0))^n M_0 + \frac{g_0}{c_0 + k_0} [(1 + k(c_0 + k_0))^n - 1],$$

and inequality (6.2) results when we set $nk = t$.

To justify (6.3), we shall consider boundary values, initial values, and difference equations pertaining to the t-difference quotients p_{ij}^n of the u_{ij}^n. From (4.2), we have

$$p_{Ij}^n = 0 \qquad \text{for} \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0, \tag{6.5}$$

and, from (4.3),

$$p_{0j}^n = p_{0,-J}^n \qquad \text{for} \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0. \tag{6.6}$$

The initial values p_{ij}^0 are to be found from the u_{ij}^0, which are given by (4.1), and the u_{ij}^1, which must be calculated. It will be enough here to know that the p_{ij}^0 are bounded and, thus, to justify the following estimate:

$$|p_{ij}^0| \leqslant 2\phi_1 + (c_0 + k_0)\phi_0 + g_0 \qquad \text{for} \quad |j| \leqslant J, \quad 0 \leqslant i \leqslant I. \tag{6.7}$$

To do so, we first rewrite Eqs. $(4.4)_-$ with $n = 0$ as

$$p_{ij}^0 + jh'q_{i+1,j}^0 + \theta' b_{ij}(p_{ij}^0 - p_{i,j-1}^0) + b_{ij}r_{ij}^0 + c_{ij}^0 u_{ij}^0 - g_{ij}^0 - S_{ij}^0 = 0,$$
$$0 \leqslant i \leqslant I - 1, \qquad -J \leqslant j \leqslant 0.$$

From this we easily see that the quantities

$$B_{ij} = p_{ij}^0 + \theta' b_{ij}(p_{ij}^0 - p_{i,j-1}^0) \tag{6.8}$$

are bounded for $0 \leqslant i \leqslant I - 1$, $-J \leqslant j \leqslant 0$. [We interpret $B_{i,-J}$ as $p_{i,-J}^0$, in accordance with our convention respecting Eq. $(4.4)_-$.] In fact, for these values of the indices, applying hypotheses $(\mathrm{i})_1$ to $(\mathrm{iv})_1$ of Section 4, we have

$$\begin{aligned} |B_{ij}| &\leqslant |j|\, h'\, |q_{i+1,j}^0| + b_{ij}\, |r_{ij}^0| + (c_0 + k_0)\phi_0 + g_0 \\ &\leqslant \phi_1(|j|\, h' + 1 - (jh')^2) + (c_0 + k_0)\phi_0 + g_0 \\ &\leqslant C, \end{aligned}$$

where

$$C = 2\phi_1 + (c_0 + k_0)\phi_0 + g_0 \, .$$

From these estimates, we can now prove that

$$| p^0_{ij} | \leqslant C \tag{6.9}$$

for $0 \leqslant i \leqslant I - 1$, $-J \leqslant j \leqslant 0$. For this purpose, we rewrite (6.8) as

$$p^0_{ij} = s_{ij} B_{ij} + (1 - s_{ij})\, p^0_{i,j-1} \, , \tag{6.10}$$

where

$$s_{ij} = (1 + \theta' b_{ij})^{-1};$$

note that $0 < s_{ij} \leqslant 1$. Since $s_{i,-J} = 1$, the foregoing estimates show, in particular, that

$$| p^0_{i,-J} | = | B_{i,-J} | \leqslant C.$$

If we assume for induction that for some j with $-J < j \leqslant 0$ we have

$$| p^0_{i,j-1} | \leqslant C \qquad \text{for} \quad 0 \leqslant i \leqslant I - 1,$$

then we can conclude from (6.10) that

$$| p^0_{ij} | \leqslant s_{ij} C + (1 - s_{ij})\, C = C.$$

It follows that (6.9) is true for $0 \leqslant i \leqslant I - 1$, $-J \leqslant j \leqslant 0$, as asserted. We shall now justify (6.9) for all other possible values of i and j. For $i = I$, $-J \leqslant j \leqslant 0$, condition (6.9) follows trivially from (6.5). For $i = 0$, $1 \leqslant j \leqslant J$, since

$$| p^0_{0j} | = | p^0_{0,-J} | \leqslant C$$

by (6.6) and a previous result, condition (6.9) again is justified. Finally consider the cases $1 \leqslant i \leqslant I$, $1 \leqslant j \leqslant J$. Arguing from Eqs. $(4.4)_+$ as we did above from $(4.4)_-$, we can easily deduce that

$$| B_{ij} | \leqslant C$$

in these cases. Equation (6.10) continues to hold, while

$$| p^0_{i0} | \leqslant C \qquad \text{for} \quad 1 \leqslant i \leqslant I$$

from previous results. Mathematical induction with respect to j now will enable us to show that, for each $i = 1, 2, \ldots, I$, condition (6.9) is true for

$j = 1, 2, \ldots, J$. In sum, condition (6.9) is seen to be justified in all cases, condition (6.7) being identical to it.

The next step in estimating p_{ij}^n is to form difference equations for these quantities by taking t-differences in Eqs. (4.4). The difference equations that result are of the same form as (4.4) with the same coefficient and kernel, but with a new inhomogeneous part involving t-differences of the original coefficient and kernel and involving the solution of the original problem. This original solution is estimated by (6.2) and the other quantities by constants stated in the hypotheses (Section 4). In view of (6.5)–(6.7), p_{ij}^n may now be estimated by the same means as was u_{ij}^n and thus shown to be bounded on S_T for any $T > 0$, as contended. We shall not carry out the details.

7. Assumption of Boundary Data

General estimates for the x-difference quotients of $u^{h,h',k}(x, y, t)$ depend upon boundary estimates of these quantities, which because of the zero boundary condition, follow from bounds for $u^{h,h',k}(x, y, t)$ itself at lattice points close to the boundary. Such bounds can be expected only in truncated problems.

The first result is independent of truncation and states that, under the hypotheses of Section 4 and condition (6.1),

$$
\begin{aligned}
&| u^{h,h',k}(x, y, t)| \leqslant e^{c_0 T} \max(\phi_1 , b/| y |)(X - x) \\
&\text{for} \quad -1 \leqslant y < 0, \quad 0 < x \leqslant X, \quad 0 \leqslant t \leqslant T,
\end{aligned} \tag{7.1}
$$

where

$$b = g_0 + k_0 M(T).$$

The second result states that, if

$$K(x, y, t, y') = g(x, y, t) = 0 \qquad \text{for} \quad | y | < \omega,$$

ω being a constant, and if the previous assumptions continue to hold, then

$$
\begin{aligned}
&| u^{h,h',k}(x, y, t)| \leqslant e^{c_0 T} \max(\phi_1 , b/\omega)(X - x) \\
&\text{for} \quad -1 \leqslant y < 0, \quad 0 < x \leqslant X, \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{7.2}
$$

It will be assumed for convenience that h' divides ω and that k divides T.

To prove these results, we first change variables in $(4.4)_-$, making the substitution

$$u_{ij}^n = (1 + c_0 k)^n \, U_{ij}^n .$$

After some manipulation, we thereby obtain

$$\begin{aligned}(1 + c_0k)(1 + \theta' b_{ij})\, U_{ij}^{n+1} &= (1 - kc_{ij}^n + \theta jh')\, U_{ij}^n - \theta jh' U_{i+1,j}^n \\ &+ \theta'(1 + c_0k)\, b_{ij} U_{i,j-1}^{n+1} + k(1 + c_0k)^{-n} g_{ij}^n + k(1 + c_0k)^{-n} S_{ij}^n \\ &\text{for} \quad 0 \leqslant i \leqslant I - 1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0. \qquad (7.3)\end{aligned}$$

To make estimations, we define

$$v_{ij} = \max_{\substack{0 \leqslant nk \leqslant T \\ -J \leqslant j' \leqslant j}} |\, U_{ij'}^n \,|.$$

From (7.3) we then have

$$\begin{aligned}(1 + c_0k)(1 + \theta' b_{ij})\, |\, U_{ij}^{n+1} \,| &\leqslant (1 + \theta jh' + kc_0)\, v_{ij} - \theta jh' v_{i+1,j} \\ &+ \theta'(1 + c_0k)\, b_{ij} v_{i,j-1} + kb \\ \text{for} \quad 0 \leqslant i \leqslant I - 1, \quad -J < j \leqslant 0,& \quad 0 \leqslant nk \leqslant T - k, \qquad (7.4)\end{aligned}$$

since

$$(1 + c_0k)^{-n} (|\, g_{ij}^n \,| + |\, S_{ij}^n \,|) \leqslant |\, g_{ij}^n \,| + |\, S_{ij}^n \,| \leqslant g_0 + k_0 M(T) = b.$$

From the special convention applying to the value $j = -J$ in Eqs. $(4.4)_-$, we have also

$$\begin{aligned}(1 + c_0k)\, |\, U_{i,-J}^{n+1} \,| &\leqslant (1 - \theta + kc_0)\, v_{i,-J} + \theta v_{i+1,-J} + kb \\ &\text{for} \quad 0 \leqslant i \leqslant I - 1, \quad 0 \leqslant nk \leqslant T - k. \qquad (7.4)'\end{aligned}$$

Consider any fixed i and j with $0 \leqslant i \leqslant I - 1$ and $-J \leqslant j < 0$. Either an index m exists such that $-J \leqslant m \leqslant j$ and

$$v_{ij} = |\, U_{im}^0 \,|, \qquad (7.5)_0$$

in which case

$$v_{ij} \leqslant \phi_1(I - i)\, h, \qquad (7.5)$$

or we have the contrary case. In the contrary case, indices n and j' exist such that $0 < (n + 1)\, k \leqslant T$, $-J \leqslant j' \leqslant j$, and

$$v_{ij} = |\, U_{ij'}^{n+1} \,|. \qquad (7.6)$$

If $j' = -J$, we deduce from (7.4)′ that

$$(1 + c_0k)\, v_{ij} = (1 + c_0k)\, |\, U_{i,-J}^{n+1} \,| \leqslant (1 + kc_0 - \theta)\, v_{i,-J} + \theta v_{i+1,-J} + kb.$$

This inequality is preserved if we replace $v_{i,-J}$ and $v_{i+1,-J}$ by v_{ij} and $v_{i+1,j}$, respectively. Then making cancellations and rearranging, we have

$$v_{ij} \leqslant v_{i+1,j} + hb \tag{7.7$'$}$$

(recall $\theta = k/h$). If (7.6) holds with $j' > -J$, however, we get from (7.4) that

$$\begin{aligned}(1 + c_0 k)(1 + \theta' b_{ij'})\, v_{ij} &= (1 + c_0 k)(1 + \theta' b_{ij'}) \mid U_{ij'}^{n+1} \mid \\ &\leqslant (1 + \theta j' h' + k c_0)\, v_{ij'} - \theta j' h' v_{i+1,j'} + \theta'(1 + c_0 k)\, b_{ij'} v_{i,j'-1} + kb.\end{aligned}$$

In this, we can replace $v_{ij'}$ by v_{ij}, $v_{i+1,j'}$ by $v_{i+1,j}$, and $v_{i,j'-1}$ by v_{ij} and then make appropriate cancellations and rearrangements to find that

$$v_{ij} \leqslant v_{i+1,j'} + \frac{hb}{\mid j' \mid h'}.$$

Since $j' \leqslant j < 0$, we thus have, finally,

$$v_{ij} \leqslant v_{i+1,j} + \frac{hb}{\mid j \mid h'}, \tag{7.7}$$

a relation that includes (7.7)$'$ and thus is valid whenever condition $(7.5)_0$ fails. If, with some j, condition (7.7) holds for all i such that $i' \leqslant i \leqslant i''$ we have by addition

$$v_{i'j} \leqslant v_{i''j} + \frac{b(i'' - i')h}{\mid j \mid h'}. \tag{7.8}$$

In this inequality, replace i' by i, and let i'' be the greatest index such that $i < i'' \leqslant I$ for which the inequality remains good. If $i'' = I$, we deduce

$$v_{ij} \leqslant \frac{b(I - i)h}{\mid j \mid h'}, \tag{7.9a}$$

since $v_{Ij} = 0$. If $i'' < I$, condition (7.5) applies, showing that

$$v_{i''j} \leqslant \phi_1(I - i'')\, h$$

and thus, in view of (7.8), that

$$v_{ij} \leqslant \phi_1(I - i'')h + \frac{b(i'' - i)h}{\mid j \mid h'}. \tag{7.9b}$$

Since either (7.9a) or (7.9b) is the case, we have

$$v_{ij} \leqslant \max_{i < i'' \leqslant I} L(i''),$$

where

$$L(i'') = \phi_1(I - i'')h + \frac{b(i'' - i)h}{|j|h'}.$$

A linear function of one variable takes its maximum for a given interval at one of the end points of the interval. Hence,

$$L(i'') \leqslant \max\left(\phi_1(I - i)h, \frac{b(I - i)h}{|j|h'}\right) \qquad \text{for} \quad i \leqslant i'' \leqslant I,$$

and, therefore,

$$v_{ij} \leqslant \max\left(\phi_1, \frac{b}{|j|h'}\right)(I - i)h.$$

This and the estimation

$$|u_{ij}^n| \leqslant (1 + c_0 k)^n v_{ij} = [(1 + c_0 k)^{1/k}]^{nk} v_{ij} \leqslant e^{c_0 T} v_{ij}$$

imply (7.1).

To prove inequality (7.2), we need consider only the values of y for which $-\omega < y < 0$, the inequality already following from (7.1) whenever $-1 \leqslant y \leqslant -\omega$. The estimations will be made in terms of the following quantities:

$$\tilde{V}_{ij} = \max_{\substack{-\omega < j'h' \leqslant jh' \\ 0 \leqslant nk \leqslant T}} |U_{ij'}^n| \qquad \text{for} \quad -\omega < jh' < 0,$$

$$V_i = e^{c_0 T} \max(\phi_1, b/\omega)(I - i)h,$$

$$V_{ij} = \max(\tilde{V}_{ij}, V_i).$$

Since $g_{ij}^n = 0$ and $S_{ij}^n = 0$ when $-\omega \leqslant jh' \leqslant 0$, we immediately have from (7.3) the estimate

$$(1 + c_0 k)(1 + \theta' b_{ij}) |U_{ij}^{n+1}| \leqslant (1 + kc_0 + \theta jh') \tilde{V}_{ij} - \theta jh' \tilde{V}_{i+1,j} + \theta'(1 + c_0 k) b_{ij} V_{i,j-1}$$

$$\text{for} \quad -\omega < jh' \leqslant 0, \quad 0 \leqslant i \leqslant I - 1, \quad 0 \leqslant nk \leqslant T - k. \qquad (7.10)$$

An analysis like that of (7.4) shows that either an index m exists such that $-\omega < mh' \leqslant jh'$ and $\tilde{V}_{ij} = |U_{im}^0|$, in which case we have

$$\tilde{V}_{ij} \leqslant \phi_1(I - i)h \leqslant V_i, \qquad (7.11a)$$

or else values n and j' exist such that $-\omega < j'h' \leqslant jh'$, $0 < (n+1)\,k \leqslant T$, and $\tilde{V}_{ij} = | \, U_{ij'}^{n+1} \, |$. In the latter case, we readily deduce from (7.10) that

$$(1 + c_0 k)\, \theta' b_{ij'} \tilde{V}_{ij} \leqslant \theta j' h' \tilde{V}_{ij} - \theta i' h' \tilde{V}_{i+1,j'} + \theta'(1 + c_0 k)\, b_{ij'} V_{i,j'-1} \,.$$

If $\tilde{V}_{ij}$ on the left of this inequality is replaced by V_i, the result is also correct. Hence, we can in fact replace $\tilde{V}_{ij}$ by V_{ij} and thus arrive at

$$(1 + c_0 k)\, \theta' b_{ij'} V_{ij} \leqslant \theta j' h' \tilde{V}_{ij} - \theta j' h' \tilde{V}_{i+1,j'} + \theta'(1 + c_0 k)\, b_{ij'} V_{i,j'-1} \,.$$

Now replacing $\tilde{V}_{i+1,j'}$ by $\tilde{V}_{i+1,j}$ and $V_{i,j'-1}$ by V_{ij}, and then making a cancellation, we obtain

$$\tilde{V}_{ij} \leqslant \tilde{V}_{i+1,j} \,. \tag{7.11b}$$

Either (7.11a) or (7.11b) must hold for $0 \leqslant i \leqslant I - 1$, $-\omega < jh' < 0$, while $V_{Ij} = 0 = V_I$ for $-\omega \leqslant jh' \leqslant 0$. We shall show that, in consequence,

$$\tilde{V}_{ij} \leqslant V_i \,, \qquad \text{for} \quad 0 \leqslant i \leqslant I, \quad -\omega < jh' < 0. \tag{7.12}$$

As we have already noted, inequalities (7.12) hold for $i = I$. Then let k denote the least integer such that $0 \leqslant k \leqslant I$ and

$$\tilde{V}_{ij} \leqslant V_i \qquad \text{for} \quad k \leqslant i \leqslant I, \quad -\omega < jh' < 0.$$

If $k > 0$, an index j_0 exists such that $-\omega < j_0 h' < 0$ and

$$\tilde{V}_{k-1,j_0} > V_{k-1} \,. \tag{7.13}$$

This excludes (7.11a) for $i = k - 1$, and we thus have by (7.11b)

$$\tilde{V}_{k-1,j_0} \leqslant \tilde{V}_{kj_0} \,.$$

By the definition of k, we have

$$\tilde{V}_{kj_0} \leqslant V_k \,,$$

while

$$V_k < V_{k-1} \,.$$

Hence

$$\tilde{V}_{k-1,j_0} \leqslant \tilde{V}_{kj_0} \leqslant V_k < V_{k-1} \,,$$

a statement contradicting (7.13). It follows that $k = 0$ and thus that (7.12) is correct, as asserted. This justifies (7.2) for $-\omega < y < 0$ and thus for $-1 \leqslant y < 0$.

8. Bounds for x- and y-Differences of Solutions of Difference Equations

Our convergence proof for truncated problems (Section 5) required knowledge of a suitable bound for $u^{h,h',k}(x, y, t)$ and of Lipschitz conditions with respect to x, y, and t holding uniformly in the region

$$S_{T,\delta} : \delta \leqslant x \leqslant X, \quad |y| \leqslant 1 - \delta, \quad 0 \leqslant t \leqslant T,$$

where δ is any positive number less than X. A bound and a Lipschitz condition with respect to t were given in Section 6 under the restriction

$$\theta + kc_0 \leqslant 1.$$

Here, we shall deduce Lipschitz conditions with respect to x and y under the stronger requirement:

$$2\theta + kc_0 \leqslant 1. \tag{8.1}$$

These Lipschitz conditions are

$$x \,|\, q^{h,h',k}(x, y, t)| \leqslant C \tag{8.2}$$

and

$$(1 - y^2)|\, r^{h,h',k}(x, y, t)| \leqslant C, \tag{8.3}$$

where C is a suitable constant independent of h, h', k, x, y, t. (While these conditions suffice for the proof of convergence, considerably stronger inequalities apply to the eventual actual solution, see [*3*].)

We shall first justify (8.2). Subtracting equations of (4.4)$_-$ for consecutive values of i and dividing by h, we obtain

$$\begin{aligned} q_{ij}^{n+1} - q_{ij}^{n} + \theta j h'(q_{i+1,j}^{n} - q_{ij}^{n}) + \theta'[b_{ij} h^{-1}(u_{ij}^{n+1} - u_{i,j-1}^{n+1}) \\ - b_{i-1,j} h^{-1}(u_{i-1,j}^{n+1} - u_{i-1,j-1}^{n+1})] + k c_{ij}^{n} q_{ij}^{n} + k h^{-1}(c_{ij}^{n} - c_{i-1,j}^{n})\, u_{i-1,j}^{n} \\ = k h^{-1}(g_{ij}^{n} - g_{i-1,j}^{n}) + k h^{-1} h' \Sigma_{j'} (K_{ijj'}^{n} - K_{i-1,jj'}^{n})\, u_{i-1,j'}^{n} + k h' \Sigma_{j'} K_{ijj'}^{n} q_{ij'}^{n} \end{aligned} \tag{8.4$_-$}$$

for

$$1 \leqslant i \leqslant I - 1, \qquad -J \leqslant j \leqslant 0, \qquad n \geqslant 0.$$

Substituting $b_{ij} = (1 - j^2h'^2)/(i+1)\,h$ in the terms involving θ' above and rewriting, gives for these terms the expression

$$\theta'(1 - j^2h'^2)\left[\frac{1}{(i+1)h}h^{-1}(u_{ij}^{n+1} - u_{i,j-1}^{n+1}) - \frac{1}{ih}h^{-1}(u_{i-1,j}^{n+1} - u_{i-1,j-1}^{n+1})\right]$$

$$= \theta'(1 - j^2h'^2)\left[\frac{1}{(i+1)h}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) + \left(\frac{1}{(i+1)h} - \frac{1}{ih}\right)h^{-1}h'r_{i-1,j}^{n+1}\right]$$

$$= \theta' b_{ij}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) - \theta b_{i-1,j}(i+1)^{-1} r_{i-1,j}^{n+1} .$$

Condition $(4.4)_-$ states, however, that

$$p_{ij}^n + jh'q_{i+1,j}^n + b_{ij}r_{ij}^{n+1} + c_{ij}^n u_{ij}^n = g_{ij}^n + S_{ij}^n .$$

Hence,

$$b_{i-1,j}r_{i-1,j}^{n+1} = -p_{i-1,j}^n - jh'q_{ij}^n - c_{i-1,j}^n u_{i-1,j}^n + g_{i-1,j}^n + S_{i-1,j}^n$$

and, by substitution, the previous expression containing θ' reduces to

$$\theta' b_{ij}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) + \frac{\theta jh'}{i+1} q_{ij}^n + \frac{\theta}{i+1} A_{ij}^n , \tag{8.5$_-$}$$

where

$$A_{ij}^n = p_{i-1,j}^n + c_{i-1,j}^n u_{i-1,j}^n - S_{i-1,j}^n - g_{i-1,j}^n$$

is a bounded quantity because of the uniform estimates for p_{ij}^n and u_{ij}^n (Section 6) and the hypotheses made concerning the coefficients (Section 4). Using this expression in $(8.4)_-$ gives

$$q_{ij}^{n+1} - q_{ij}^n + \theta jh'\left(q_{i+1,j}^n - q_{ij}^n + \frac{1}{i+1} q_{ij}^n\right) + \theta' b_{ij}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) + kc_{ij}^n q_{ij}^n$$

$$= G_{ij}^n + kh'\Sigma_{j'} K_{ijj'}^n q_{ij'}^n \tag{8.6$_-$}$$

for $1 \leqslant i \leqslant I - 1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0,$

where

$$G_{ij}^n = -\frac{\theta}{i+1} A_{ij}^n - ku_{i-1,j}^n h^{-1}(c_{ij}^n - c_{i-1,j}^n)$$
$$+ kh^{-1}(g_{ij}^n - g_{i-1,j}^n) + kh^{-1}h'\Sigma_{j'}(K_{ijj'}^n - K_{i-1,jj'}^n)\, u_{i-1,j'}^n .$$

(Recall our convention under which any expression containing $b_{i,-J}$ as a factor is to be replaced by zero.) By our assumptions as to Lipschitz continuity (Section 4), a constant G exists such that

$$(i+1)h \mid G_{ij}^n \mid \leqslant kG \quad \text{for} \quad 1 \leqslant i \leqslant I-1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0. \quad (8.7)_-$$

We shall now obtain equations for q_{ij}^n with $1 \leqslant j \leqslant J$, to this end subtracting equations of $(4.4)_+$ for consecutive values of i. Instead of using $(8.5)_-$, we here rewrite the expression

$$\theta'[b_{ij}h^{-1}(u_{ij}^{n+1} - u_{i,j-1}^{n+1}) - b_{i-1,j}h^{-1}(u_{i-1,j}^{n+1} - u_{i-1,j-1}^{n+1})],$$

which occurs again, as

$$\theta' b_{i-1,j}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) - \frac{\theta}{i} b_{ij} r_{ij}^{n+1}$$

and substitute for $b_{ij}r_{ij}^{n+1}$ from $(4.4)_+$ in the form

$$p_{ij}^n + jh'q_{ij}^n + b_{ij}r_{ij}^{n+1} + c_{ij}^n u_{ij}^n = g_{ij}^n + S_{ij}^n$$

$$1 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0.$$

In this way, we obtain

$$q_{ij}^{n+1} - q_{ij}^n + \theta jh' \left(q_{ij}^n - q_{i-1,j}^n + \frac{1}{i} q_{ij}^n\right) + \theta' b_{i-1,j}(q_{ij}^{n+1} - q_{i,j-1}^{n+1}) + kc_{ij}^n q_{ij}^n$$

$$= G_{ij}^n + kh'\Sigma_{j'} K_{ijj'}^n q_{ij'}^n, \quad (8.6)_+$$

where

$$G_{ij}^n = -\frac{\theta}{i}(p_{ij}^n + c_{ij}^n u_{ij}^n - g_{ij}^n - S_{ij}^n)$$

$$+ k[-u_{i-1,j}^n h^{-1}(c_{ij}^n - c_{i-1,j}^n) + h^{-1}(g_{ij}^n - g_{i-1,j}^n)]$$

$$+ kh^{-1}h'\Sigma_{j'}(K_{ijj'}^n - K_{i-1,jj'}^n)\, u_{i-1,j'}^n,$$

$$2 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0.$$

Under our assumptions,

$$ih \mid G_{ij}^n \mid \leqslant kG \quad \text{for} \quad 2 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0 \quad (8.7)_+$$

if G is made sufficiently large.

We now change variables, introducing

$$Q_{ij}^n = 2ihq_{ij}^n \qquad \text{for} \quad 1 \leqslant i \leqslant I, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0,$$

$$= (i+1)\, hq_{ij}^n \qquad \text{for} \quad 1 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0.$$

This change of variables is useful, first, because

$$Q^n_{i+1,j} - Q^n_{ij} = 2(i+1)h\left[q^n_{i+1,j} - q^n_{ij} + \frac{1}{i+1}\, q^n_{ij}\right]$$

$$\text{for} \quad 1 \leqslant i \leqslant I-1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0, \tag{8.8$_-$}$$

and

$$Q^n_{ij} - Q^n_{i-1,j} = ih\left[q^n_{ij} - q^n_{i-1,j} + \frac{1}{i}\, q^n_{ij}\right]$$

$$\text{for} \quad 2 \leqslant i \leqslant I, \quad 1 \leqslant j \leqslant J, \quad n \geqslant 0. \tag{8.8$_+$}$$

Secondly,

$$2ih(q^n_{ij} - q^n_{i,j-1}) = Q^n_{ij} - Q^n_{i,j-1}$$

$$\text{for} \quad 1 \leqslant i \leqslant I, \quad 1-J \leqslant j \leqslant 0, \quad n \geqslant 0, \tag{8.9$_-$}$$

$$(i+1)\, h(q^n_{ij} - q^n_{i,j-1}) = Q^n_{ij} - Q^n_{i,j-1}$$

$$\text{for} \quad 1 \leqslant i \leqslant I, \quad 2 \leqslant j \leqslant J, \quad n \geqslant 0, \tag{8.9$_+$}$$

and

$$(i+1)\, h(q^n_{i1} - q^n_{i0}) = Q^n_{i1} - \frac{i+1}{2i}\, Q^n_{i0}$$

$$\text{for} \quad 1 \leqslant i \leqslant I, \quad n \geqslant 0. \tag{8.9$_0$}$$

The form particularly of $(8.9)_0$ is crucial in the estimation process below.

Let us now make the indicated change of variables in Eqs. (8.6) Multiplying Eqs. $(8.6)_-$ by $2ih$, multiplying Eqs. $(8.6)_+$ by $(i+1)\,h$, substituting for the q's, using (8.8) and (8.9), and solving for Q^{n+1}_{ij} give us equivalent relations. With

$$K^n_{ijj'}{}' = K^n_{ijj'} \qquad \text{for} \quad \begin{Bmatrix} -J \leqslant j \leqslant 0 \\ -J \leqslant j' \leqslant 0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} 1 \leqslant j \leqslant J \\ 1 \leqslant j' \leqslant J \end{Bmatrix}$$

$$= \frac{2i}{i+1}\, K^n_{ijj'} \qquad \text{for} \quad \begin{Bmatrix} -J \leqslant j \leqslant 0 \\ 1 \leqslant j' \leqslant J \end{Bmatrix}$$

$$= \frac{i+1}{2i}\, K^n_{ijj'} \qquad \text{for} \quad \begin{Bmatrix} 1 \leqslant j \leqslant J \\ -J \leqslant j' \leqslant 0 \end{Bmatrix},$$

we write these results as follows:

$$(1 + \theta' b_{ij})\, Q^{n+1}_{ij} = \left(1 + \theta jh' \left(\frac{i}{i+1}\right) - kc^n_{ij}\right) Q^n_{ij} - \theta jh' \left(\frac{i}{i+1}\right) Q^n_{i+1,j}$$

$$+ \theta' b_{ij} Q^{n+1}_{i,j-1} + kh' \Sigma_{j'} K^n_{ijj'}{}' Q^n_{ij'} + 2ihG^n_{ij}$$

$$\text{for} \quad 1 \leqslant i \leqslant I-1, \quad -J \leqslant j \leqslant 0, \quad n \geqslant 0, \tag{8.10$_-$}$$

$$(1 + \theta' b_{i-1,j}) Q_{ij}^{n+1} = \left(1 - \theta j h' \left(\frac{i+1}{i}\right) - k c_{ij}^n\right) Q_{ij}^n + \theta j h' \left(\frac{i+1}{i}\right) Q_{i-1,j}^n$$
$$+ \theta' b_{i-1,j} Q_{i,j-1}^{n+1} + k h' \Sigma_{j'} K_{ijj'}^n{}' Q_{ij'}^n + (i+1)\, h G_{ij}^n$$
$$\text{for} \quad 2 \leqslant i \leqslant I, \quad 2 \leqslant j \leqslant J, \quad n \geqslant 0, \quad (8.10)_+$$

and

$$+ \theta' b_{i-1,1}) Q_{i1}^{n+1} = \left(1 - \theta h' \left(\frac{i+1}{i}\right) - k c_{i1}^n\right) Q_{i1}^n + \theta h' \left(\frac{i+1}{i}\right) Q_{i-1,1}^n$$
$$+ \theta' b_{i-1,1} \left(\frac{i+1}{2i}\right) Q_{i0}^{n+1}$$
$$+ k h' \Sigma_{j'} K_{i1j'}^n{}' Q_{ij'}^n + (i+1)\, h G_{i1}^n$$
$$\text{for} \quad 2 \leqslant i \leqslant I, \quad n \geqslant 0. \quad (8.10)_0$$

We now make several remarks preparatory to estimating the Q's. First, assumption (8.1) ensures that the coefficient of Q_{ij}^n on the right side of each one of Eqs. (8.10) is nonnegative. Secondly,

$$| K_{ijj'}^n{}' | \leqslant 2 \, | K_{ijj'}^n | \qquad \text{for} \quad 1 \leqslant i \leqslant I.$$

Thirdly, we recall estimates (8.7). Finally, bounds exist for

$$\begin{aligned} &Q_{1j}^{n+1}, \quad 1 \leqslant j \leqslant J, \qquad 0 \leqslant nk \leqslant T - k, \\ &Q_{Ij}^{n+1}, \ -J \leqslant j \leqslant 0, \qquad 0 \leqslant nk \leqslant T - k, \end{aligned} \tag{8.11}$$

which are the only Q_{ij}^{n+1} occurring in the right but not the left member of one of the Eqs. (8.10). In fact,

$$\begin{aligned} | Q_{1j}^n | &= 2h \, | q_{1j}^n | = 2 \, | u_{1j}^n - u_{0j}^n | \\ &\leqslant 4M(T) \qquad \text{for} \quad 0 \leqslant nk \leqslant T, \end{aligned}$$

and

$$\begin{aligned} | Q_{Ij}^n | &= 2Ih \, | u_{Ij}^n - u_{I-1,j}^n | \, h^{-1} \\ &\leqslant 2X e^{c_0 T} \max(\phi_1, b/\omega) \qquad \text{for} \quad -J \leqslant j \leqslant 0 \end{aligned}$$

by (7.2) and (4.2).

From these remarks and Eqs. (8.10), we can now estimate the Q's. Define

$$Q_n{}' = \max | Q_{ij}^n |,$$

the maximum being taken for all indices i and j such that

$$1 \leqslant i \leqslant I-1, \qquad -J \leqslant j \leqslant 0$$

or

$$2 \leqslant i \leqslant I, \qquad 1 \leqslant j \leqslant J.$$

Also define

$$Q_n = \max(Q_n', 4M(T), 2Xe^{c_0 T} \max(\phi_1, b/\omega)).$$

By the previous remarks,

$$| Q_{ij}^n | \leqslant Q_n$$

for $0 \leqslant nk \leqslant T$ and all possible values of i and j; in addition, the coefficients (possibly excluding the $K_{ijj'}^n{}'$) in the right members of Eqs. (8.10) are nonnegative. From $(8.10)_-$ and $(8.7)_-$ we thus have, in particular,

$$| Q_{i-J}^{n+1} | \leqslant [1 + k(c_0 + 2k_0)] Q_n + 2kG \qquad \text{for} \quad 1 \leqslant i \leqslant I-1 \tag{8.12$_{-J}$}$$

and

$$(1 + \theta' b_{ij}) | Q_{ij}^{n+1} | \leqslant [1 + k(c_0 + 2k_0)] Q_n + \theta' b_{ij} Q_{n+1}' + 2kG \qquad \text{for} \quad 1 \leqslant i \leqslant I-1, \quad 1-J \leqslant j \leqslant 0. \tag{8.12$_-$}$$

From $(8.10)_+$ and $(8.7)_+$ we have

$$(1 + \theta' b_{i-1,j}) | Q_{ij}^{n+1} | \leqslant [1 + k(c_0 + 2k_0)] Q_n + \theta' b_{i-1,j} Q_{n+1}' + 2kG \qquad \text{for} \quad 2 \leqslant i \leqslant I, \quad 2 \leqslant j \leqslant J, \tag{8.12$_+$}$$

and from $(8.10)_0$,

$$(1 + \theta' b_{i-1,1}) | Q_{i1}^{n+1} | \leqslant [1 + k(c_0 + 2k_0)] Q_n + \theta' b_{i-1,1} Q_{n+1}' + 2kG \qquad \text{for} \quad 2 \leqslant i \leqslant I. \tag{8.12$_0$}$$

From these inequalities we have

$$Q_{n+1}' \leqslant [1 + k(c_0 + 2k_0)] Q_n + 2kG. \tag{8.13}$$

To deduce this, let i_0, j_0 denote values of i and j, respectively, for which the quantities $| Q_{ij}^{n+1} |$ occurring in the left members of inequalities (8.12) attain their maximum. Then

$$| Q_{i_0 j_0}^{n+1} | = Q_{n+1}'$$

and either

$$1 \leqslant i_0 \leqslant I - 1, \qquad -J \leqslant j_0 \leqslant 0$$

or

$$2 \leqslant i_0 \leqslant I, \qquad 1 \leqslant j_0 \leqslant J.$$

If $j_0 = -J$, we have (8.13) from $(8.12)_{-J}$. If $1 - J \leqslant j_0 \leqslant 0$, inequality. $(8.12)_-$ implies that

$$(1 + \theta' b_{i_0 j_0}) Q_{n+1}' \leqslant [1 + k(c_0 + 2k_0)] Q_n + \theta' b_{i_0 j_0} Q_{n+1}' + 2kG,$$

which implies (8.13). If $2 \leqslant j_0 \leqslant J$, we obtain (8.13) similarly by substitution into $(8.12)_+$ and, if $j_0 = 1$, by substitution into $(8.12)_0$. Inequality (8.13) therefore holds in every case, as asserted.

Inequality (8.13) implies

$$Q_{n+1} \leqslant [1 + k(c_0 + 2k_0)] Q_n + 2kG. \tag{8.14}$$

In fact, $4M(T)$ and $2Xe^{c_0 T} \max(\phi_1, b/\omega)$ are by definition not greater than Q_n and hence are not greater than the right member of (8.14). As (8.13) shows, Q_{n+1}' also is not greater than the right member of (8.14). Hence, the maximum of these three quantities, namely Q_{n+1}, is not greater than the right member of (8.14), as asserted.

Inequality (8.14) implies through mathematical induction that

$$\begin{aligned} Q_n &\leqslant Q_0[1 + k(c_0 + 2k_0)]^n + \frac{2G}{c_0 + 2k_0}\{[1 + k(c_0 + 2k_0)]^n - 1\} \\ &\leqslant Q_0 e^{(c_0+2k_0)nk} + \frac{2G}{c_0 + 2k_0}\{e^{(c_0+2k_0)nk} - 1\}. \end{aligned}$$

This gives a uniform bound for Q_{ij}^n, justifying inequality (8.2).

Inequalities (8.2), (6.2), and (6.3) give us bounds under the stated conditions for

$$| u_{ij}^n |, \quad | u_{ij}^{n+1} - u_{ij}^n |/k, \quad ih \, | u_{i+1,j}^n - u_{ij}^n |/h.$$

A corresponding bound for

$$(1 - j^2 h'^2) \, | u_{ij}^n - u_{i,j-1}^n |/h'$$

results from these on considering Eqs. (4.4) multiplied by ih/k. Thereby, inequality (8.3) is established.

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